

WHEN GAMBLING IS NOT WINNING: EXPLORING OPTIMALITY OF VIX TRADING UNDER THE EXPECTED UTILITY THEORY¹

ANMAR AL WAKIL

*University Paris-Dauphine, PSL Research University,
CNRS, DRM, 75016 Paris, France.*

We would like to thank Vikas Agarwal, Serge Darolles, Marcin Kacperczyk, Mathieu Vaissie, and Ronnie Sadka for helpful comments and suggestions. We also appreciate the comments of conference participants at the XVIth OxMetrics Conference (2015), and at the University of Paris-Dauphine workshop (2015). Any remaining errors are my responsibility.

<https://doi.org/10.26870/jbafp.2018.01.004>

SUMMARY: I. INTRODUCTION. II. DATA. III. EMPIRICAL METHODOLOGY. 1. FRAMEWORK. 2. WELFARE CRITERION MEASURE. 3. MODEL-IMPLIED RISK PREMIUM. IV. EMPIRICAL RESULTS. 1. RISK-ADJUSTED PERFORMANCE MEASURES. 2. WELFARE CRITERION MEASURE. 3. MODEL-IMPLIED RISK PREMIUM. V. CONCLUSIONS. VI. REFERENCES. VII. APPENDIX.

1. Totaling an average daily trading volume of 323,761 futures contracts. Source: CBOE Futures Exchange (CFE), as of November 3, 2014.

Abstract

Recently, financial innovations have given rise to complex derivatives within the asset management industry. Although traditional assets pay dividends or coupons, VIX futures contracts have been partly misunderstood by unsophisticated investors, as they only provide portfolio insurance against stock market crashes. Therefore, over the calmer period 2009-2014, the most traded VIX futures exchange-traded product lost practically all of its value, ruining unexperienced investors. Hence, this paper investigates appropriateness of these complex derivatives with investor's risk aversion. We address portfolio-choice optimality under uncertainty, for overlay allocations composed of equities, bonds, and VIX futures. This paper proposes a non-trivial solution based on the expected utility theory to simulate investor's behavior with risk aversion. Furthermore, it derives an investor's surprise metric defined as a welfare criterion measure, and a model-implied risk premium defined as the insurance premium investor pays ex post to hedge. Empirical results show investing in VIX futures significantly beats traditionally diversified portfolios, but they turn to be particularly inappropriate for risk-loving investors. From the asset management perspective, this paper has practical implications since it recommends pedagogical efforts to raise investors' awareness of overlay strategies.

Keywords

VIX futures; expected utility theory; resampling; portfolioallocation; performance measures; investor expectations; model-implied risk premium.

JEL Classification: C14, D81, G11, G12.

I. INTRODUCTION

*«You don't gamble to win. You gamble so you can gamble the next day.»
Bert Ambrose, English bandleader and violonist*

In the recent years, a multitude of financial innovations designed for a wide variety of investors have flourished within the asset management industry. Learning from the past, risks inherent in complex new financial products prove to be partly misunderstood, especially by unsophisticated investors. For example, this had been the case with regard to the monetization risk associated to capital protection funds, such as constant proportion portfolio insurance (CPPI), especially when risky assets underperform at launch. Similarly, this paper investigates the inherent risks in newly launched complex hedging strategies based on volatility derivatives. Specifically, developed by the Chicago Board Options Exchange (CBOE) in 2004, VIX futures contracts

have set an all-time monthly trading volume record² in October 2014, extolling the diversification effects of implied equity volatility. Following the seminal papers of Whaley (1993, 2009), volatility derivatives are assumed to provide portfolio diversification and risk-reduction, capturing the leverage effect, i.e. inverse correlation between stock market index and market volatility. In particular, Szado (2009) exhibits the portfolio insurance provided by a buy-and-hold VIX futures exposition during the subprime crisis. From August 2008 to December 2008, adding 10% VIX futures contracts to an equity-bond portfolio improves annualized return from -15.9% to -0.3% , and mitigates annualized standard deviation from 21.7% to 13.3% . More generally, Chen et al. (2011) apply mean-variance spanning tests over the period 1996-2008 to exhibit enhanced in-sample Sharpe ratios when diversifying traditional portfolios with implied equity volatility.

However, over the period from January 2009 to July 2014, the most traded short-term VIX futures exchange-traded product in the U.S., i.e. the VXX ETP, lost practically all of its value (-99.6%). This outstandingly disappointing performance brought ruin upon many inexperienced and unskilled investors, putting into question the benefits of such innovations within the asset management industry. In particular, Whaley (2013) examines ETPs benchmarked to VIX short-term futures indexes as buy-and-hold investments. From March 2004 to March 2012, he investigates the slope of VIX futures term structure at 30 day to expiration. As costs of carry prove to be painful for rollovers, he finds the average slope at 30 days to expiration is 2.3% over the period 2004-2012, and the prices curve is usually upward-sloping in nearly 81% of trading days. Consequently, his undermining intuition suggests that futures contracts in VIX Index prove to be inappropriate buy-and-hold instruments for risk-loving investors, as these instruments lose money with certainty through time. This arises from the fact that volatility derivatives do not deliver certain cash flows, forming therefore a proper asset class with specific properties. Although traditional asset classes, i.e. equities and bonds, pay either certain dividends or coupons, VIX futures rather consist in an insurance premium that investors consent to pay to hedge their portfolios against scarce stock market crashes.

In this paper, we evaluate the appropriateness of complex volatility derivatives with investor's risk-aversion degree. For this purpose, we examine the optimality of portfolio choice under uncertainty, for an overlay allocation composed of equities, bonds, and VIX futures contracts, from December 31, 2004 to July 4, 2014. Therefore, the issue this paper addresses is not trivial, as the approaches commonly used within the

2. Totaling an average daily trading volume of 323,761 futures contracts. Source: CBOE Futures Exchange (CFE), as of November 3, 2014.

asset management industry and the existing literature are inappropriate here. On the one hand, Alexander and Korovilas (2011) apply the standard mean-variance criterion to examine the diversification effects provided by buy-and-hold investments in VIX futures. However, as described by Jondeau and Rockinger (2006), this common framework pioneered by Markowitz (1959) inappropriately handles complex derivatives, as it proves ineffective under large departure from normality. Featured by strongly non-normal return distributions, VIX futures contracts require to investigate portfolio optimality under non-quadratic preferences to take into account higher-order moments. On the other hand, the practical issue usually met by asset managers when implementing optimal portfolio strategies consists in mitigating the frequency of portfolio rebalancing, costly for investors. However, this information loss proves to be statistically detrimental for portfolio optimization.

This paper investigates optimality of investment decisions under uncertainty by modeling investor's behavior under the normative decision theory. Inspired by the Expected Utility (EU) theory à la von Neumann-Morgenstern (1947) and by seminal papers pioneered by Samuelson (1969) and Merton (1969), we perform direct numerical optimizations to maximize the EU of investor's terminal wealth. Direct optimization relies specifically on the empirical estimation of joint returns distributions, based on a multivariate block bootstrap procedure described by Kunsch (1989) to capture the dependence structure of neighbored data, both over time and cross-sectionally. This approach, ensuring robustness among alternative time-settings, consistently addresses the non-quadratic preferences entailed by strongly non-Gaussian returns on VIX futures. Therefore, we implement an asset-allocation strategy for a portfolio composed of equities, bonds, and VIX futures. We address investment-decision optimality under three various criterion measures. As a first step, optimal portfolios are examined under the criterion of risk-adjusted performance measures, especially the Adjusted for Skewness Sharpe ratio (*ASSR*) introduced by Koekebakker and Zakamouline (2009), handling risk-preferences at the third order. On the one hand, between traditional portfolios, portfolios composed of equities and bonds, and alternative portfolios, i.e. portfolios diversified with VIX futures. On the other hand, between alternative portfolios depending on investor's risk aversion. As a second step, we evaluate the investor welfare gains provided by VIX futures optimal positioning, particularly depending on investor's risk appetite. Inspired by the microeconomics works pioneered by Akerlof and Dickens (1982), we derive an investor's surprise metric, defined as a welfare criterion measure. More precisely, investors feel satisfaction as the final outcome they obtain ex post exceeds their rational expectations. Conversely, negative investor's surprise corresponds to unfulfilled

expectations, generating ex post investor's pain. As a third step, this paper addresses optimality of portfolio insurance by extracting the model-implied risk premium from optimal portfolios. This defines the insurance premium that the rational investor implicitly consented to pay ex post in order to hedge his portfolio against extreme events.

Empirical results provide the three following evidence that proved to be robust, both in-sample and when implementing portfolio strategies, and whatever the time settings. First, under the criteria of risk-adjusted performance measures and investor welfare, investing in VIX futures significantly beats traditionally diversified allocations, across the relative risk-aversion coefficient γ . In-sample *ASSR* exhibits that portfolios diversified with VIX futures (4.09) significantly outperforms equity-bond portfolios (1.93), when $\gamma = 7$ for example. Moreover, implemented strategies preserve the notable outperformance of alternatively diversified portfolios (0.24) against traditional portfolios (0.14). Therefore, VIX futures positioning significantly improves the ex post investor welfare, whatever the risk aversion. For example, when $\gamma = 5$, ex post positive surprise is on average 47% higher for portfolios adding VIX futures, suggesting that they significantly exceed investor rational expectations. Second, empirical results confirm that VIX futures contracts are particularly inappropriate buy-and-hold instruments for risk-loving investors. Increasing the relative risk-aversion coefficient from Increasing the relative risk-aversion coefficient from $\gamma = 2$ to $\gamma = 12$ efficiently improves in level our investor welfare metric from 0.17% to 0.51%. This suggests that, when investing in VIX futures, risk-loving investors tend to feel more ex post pain than risk-averse investors. For example, when risk aversion is low for $\gamma = 2$, VIX futures positioning provides notably higher investor disappointment (-1.35%) than traditional asset classes (-0.75%). Besides, higher risk aversion drastically mitigates the volatility of investor surprise and of ex post discomfort, respectively by 32% and 45%. This consistently validates that risk-loving investors inappropriately evaluate the risks inherent in VIX futures contracts. Distorted by gambling attitudes, their decision process underestimates the painful costs of carry that are paid to maintain portfolio insurance. Third, the model-implied insurance premia extracted ex post from optimal portfolios are relevant with the first empirical findings. The ex post risk premia derived from alternative portfolios, i.e. equity-bonds portfolios diversified with VIX futures, significantly outdo those derived from traditional equity-bonds portfolios. This result proves to be consistent whatever the investor's risk aversion, under the EU framework. For example, when $\gamma = 3$, VIX futures optimal positioning provides far more effective insurance premium (23.13%) than traditional equity-bonds portfolios (8.21%). This result confirms that VIX futures provide better portfolio insurance against tail risks than traditional diversification.

This paper extends the existing literature in the four following ways. The technology undermining our paper upgrades the previous works of Szado (2009), Chen et al. (2011), and Alexander and Korovilas (2011). The commonly used mean-variance criterion they apply proves to be inappropriate to address portfolio choice optimality when handling derivatives, such as VIX futures contracts. Therefore, we address optimality by performing direct numerical optimizations of agent's EU. It results in handling appropriately non-quadratic agent's preferences, and in mitigating portfolio rebalancing frequency. Therefore, under this theoretical framework, we derive the model-implied risk premium from optimal positioning. This defines the portfolio insurance provided either by traditional asset classes, i.e. equities and bonds, or by VIX futures positioning. Subsequently, when agents are expected-utility maximizers, this paper extends portfolio choice optimality by exploring VIX futures positioning. To the best of our knowledge, this paper is the first examining optimality under the EU framework, for an asset allocation composed of equities, bonds, and VIX futures contracts. Sharpe (2007) maximizes EU within an asset-allocation composed of equities, bonds, and cash. Similarly, Carr and Madan (2001) study optimal positioning of European-style options under the EU theory, within a bond-equity portfolio, but they do not consider volatility derivatives. Besides, this paper illustrates the microeconomics theory pioneered by Akerlof and Dickens (1982). As decision-makers feel ex post pain if the final outcome does not exceed their rational expectations about future, we propose an original welfare criterion measure to investigate decision-process optimality under uncertainty. More precisely, this investor surprise metric evaluates the welfare gains, i.e. either positive or negative surprise, provided either by traditional asset classes, or by VIX futures positioning. Furthermore, our most decisive contribution consists in validating the intuition undermining Whaley (2013), both by risk-adjusted portfolio performance measures and by our welfare criterion metric. Empirical results confirm that VIX futures contracts are particularly inappropriate buy-and-hold investments for risk-loving investors. Distorted by gambling attitudes, risk-loving investors do not evaluate appropriately the risks inherent in VIX futures, especially the painful costs of carry.

This paper arises two practical implications, especially within the asset management industry. First, from the perspective of product management, promoting overlay and hedging strategies based on volatility derivatives requires to implement intensive pedagogical efforts to raise investors' awareness of the risks inherent to such complex derivatives. In effect, our empirical evidence proves that VIX futures contracts are particularly inappropriate buy-and-hold investments for risk-loving investors. Therefore, pedagogical efforts should bring to investors the relevant expertise to efficiently benefit from portfolio insurance provided by VIX futures

positioning. Second, from the perspective of quantitative asset managers, this paper proposes a consistent alternative approach to the commonly used mean-variance criterion. A direct numerical optimization of the expected utility appropriately handles the non-quadratic preferences related to complex derivative instruments, generalizing the mean-variance framework at higher order-moments. Furthermore, we propose two relevant metrics to examine portfolio choice optimality and portfolio insurance. The welfare criterion measure evaluates model-risk management, and the model-implied risk premium gauges portfolio risk-reduction.

The remainder of the paper is organized as follows. In section 1, we explore the dataset and discuss the statistical properties of asset returns, validating the approach undermining this paper. In section 2, we describe the standard EU asset-allocation problem and its practical implementation within the asset management industry. This section especially defines two criterion measures used to investigate portfolio-choice optimality. First, our investor welfare criterion measure gauges ex post discrepancies between the realized and expected utilities derived from VIX futures positioning. Second, the model-implied risk premium, extracted from optimal portfolios under the EU theory, evaluates portfolio insurance provided either by traditional asset classes or by VIX futures. In section 3, we investigate the empirical patterns related to portfolio-choice optimality, under the three following performance criteria: risk-adjusted portfolio performance measures, welfare criterion metric, and model-implied risk premium. Section 4 exposes some concluding remarks and practical implications within the asset management industry.

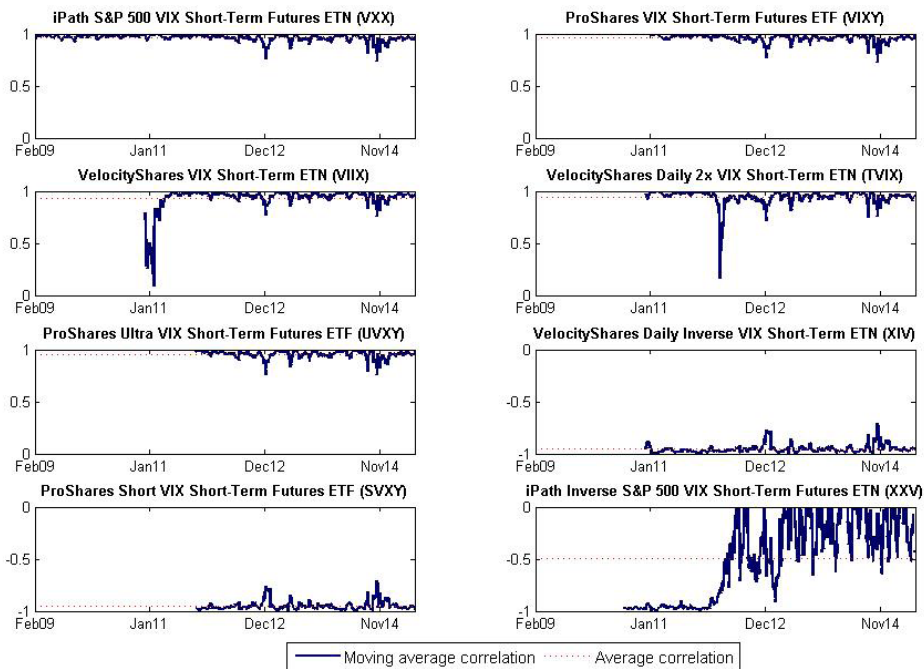
II. DATA

The dataset consists of three time series for equity, bond, and VIX futures indices, composed of 2,246 historical daily closing prices. Data sample is provided by Bloomberg, over the period from December 30, 2005 to July 4, 2014. Equity, bonds, and VIX futures indices are respectively S&P 500 Total Return Index, JPM Global Aggregate Bond Index, and S&P 500 VIX Short-Term Futures Index. The last index replicates a buy-and-hold strategy that rolls over VIX futures contracts, on a daily basis, from the nearest month to the next month. This results in maintaining a constant one-month rolling long position in the first and second month VIX futures contracts. In the previous literature, the S&P 500 VIX Short-Term Futures Index has been well documented, especially by Whaley (2013). As of March 30, 2012, seven of the eight largest VIX ETPs traded in the U.S. are benchmarked to the S&P 500 VIX Short-Term Futures Index, totalling an asset value of nearly \$2,985 million.

Figure 1 displays the time-varying Pearson correlations of the most traded VIX ETNs in the U.S. with their benchmark, the S&P 500 VIX Short-

Term Futures Index, since their inception. For multiple equal to 1, the VXX, VIXY, and VIIX ETNs have average strong positive correlations, respectively equal to 96.3%, 95.8%, and 93.0%. For leveraged ETNs, the TVIX and UVXY are also strongly positively correlated, respectively at 94.2% and 95.4%. However, much less traded VIX ETNs, like the XXV, exhibit weaker time-varying correlations. As the main VIX ETNs tend to be strongly correlated to their associated benchmark, the S&P 500 VIX Short-Term Futures Index proves to be fairly-typical of the widely traded VIX ETNs. Alternatively, our empirical study could be declined with mid-term VIX futures, rebalanced daily to maintain five-month constant maturity. Launched on March 26, 2004 by the CBOE, VIX futures contracts are preeminently characterized by a usual upward-sloping term structure, generating important costs of carry for buy-and-hold strategies. More precisely, Whaley (2013) calculates that the average slope of VIX futures term structure at 30 day to expiration is 2.3%. In other words, the 30-day futures price tends to decrease on average by 2.3% per day.

Figure 1: Correlations Between VIX ETNs with their Benchmark



This figure displays the moving average correlations between the most traded VIX ETNs and their benchmark, the S&P 500 VIX Short-Term

Futures Index, since their inception. Computations are based on the 20-day rolling Pearson correlations. Multiplier is either 1 for the VXX, VIXY, and VIIX ETNs; or 2 for the TVIX and UVXY ETNs; or -1 for the XIV, SVXY, and XXV ETNs.

Table 1 (Panel A) exhibits the outstandingly disappointing performance of VIX futures investing. This puts into question the contribution of such alternative asset within the asset management industry. From 2005 to 2014, a buy-and-hold investment in VIX futures contracts lost practically all of its value (-99.2%), and displayed a considerable annualized volatility (61.3%). In contrast, traditional asset classes such as equity or bonds achieved impressive annualized returns (respectively 9.7% and 6.5%), with much lower annualized volatilities (respectively 21.2% and 5.3%). However, breaking down the dataset into sub-periods of stock market crises (Panel B) and of calm periods (Panel C) extols the benefits of VIX futures investing for portfolio diversification and risk reduction. Although time slicing proves to be artificial, especially by violating path-dependency of asset returns, this method exhibits stylized effects characterizing these asset classes. Triggered by the Lehman Brothers bankruptcy, the subprime crisis ranges from August 29, 2008 to November 20, 2008, as VIX index spiked from 20.7% to 80.9%. Therefore, the European sovereign debt crisis ranges from July 11, 2011 to October 3, 2011, whilst the “gauge fear” index spiked from 18.4% to 45.5%. Over periods of stock market turbulence (Panel B), equities achieved negative holding period returns (-52.2%), strongly contrasting with VIX futures (1139.0%). This illustrates clearly the diversification effects exhibited by Szado (2009) of a buy-and-hold VIX futures exposition during the recent financial crises, by capturing the implied leverage effect between a stock index and its implied volatility. Following Table 2, the negative correlation between equities and VIX futures particularly increases during the periods of financial crises (-83%), whilst bonds offered only limited diversification (-17%). This illustrates the previous works of Whaley (1993, 2009) that investing in volatility derivatives could benefit to long equity investors.

Table 1: Descriptive Statistics of Asset Returns

	Daily Returns		
	Equity	Bonds	VIX Futures
Panel A: All observations			
Nb of observations	2246	2246	2246
Mean	0,04%	0,02%	- 0,14%
Median	0,06%	0,00%	- 0,43%
Standard deviation	1,34%	0,33%	3,86%
Annualized standard deviation	21,19%	5,28%	61,30%
Holding period return	86,32%	57,55%	- 99,15%
Annualized return	9,68%	6,46%	- 11,12%
Skewness	- 0,08	0,29	0,85
Kurtosis	13,78	8,41	6,94
Jarque-Bera statistic	10822,34***	2757,78***	1714,22***
Panel B: Periods of stock market crises			
Nb of observations	121	121	121
Mean	- 0,55%	- 0,01%	2,28%
Median	- 0,41%	0,00%	2,17%
Standard deviation	3,33%	0,38%	6,05%
Annualized standard deviation	52,90%	6,10%	95,98%
Holding period return	- 52,15%	- 1,47%	1139,01%
Annualized return	- 108,62%	- 3,05%	2372,14%
Skewness	0,39	- 0,72	0,40
Kurtosis	4,86	6,20	3,93
Jarque-Bera statistic	18,24***	56,18***	6,64**
Panel C: Periods of stock market calm			
Nb of observations	2125	2125	2125
Mean	0,07%	0,02%	- 0,28%
Median	0,07%	0,00%	- 0,48%
Standard deviation	1,11%	0,33%	3,65%
Annualized standard deviation	17,65%	5,23%	57,99%
Holding period return	289,41%	59,89%	- 99,93%
Annualized return	34,32%	7,10%	- 11,85%
Skewness	0,00	0,38	0,75
Kurtosis	9,20	8,58	6,96
Jarque-Bera statistic	3389,71***	2790,4***	1581,9***

This table reports the descriptive statistics of historical asset returns, from December 30, 2005 to July 4, 2014. Calculations above are based on daily simple asset returns. Stock market crises are identified as periods of stock market turmoil, ranging from August 29, 2008 to November 21, 2008, i.e. subprime crisis, and from July 11, 2011 to October 5, 2011, i.e. European sovereign debt crisis. Jarque-Bera statistic tests for the rejection of the null hypothesis, i.e. returns normality. Stars *, ** and *** denote statistical significance at the 10%, 5% and 1% level of confidence, respectively.

Table 2: Correlations between Asset Returns

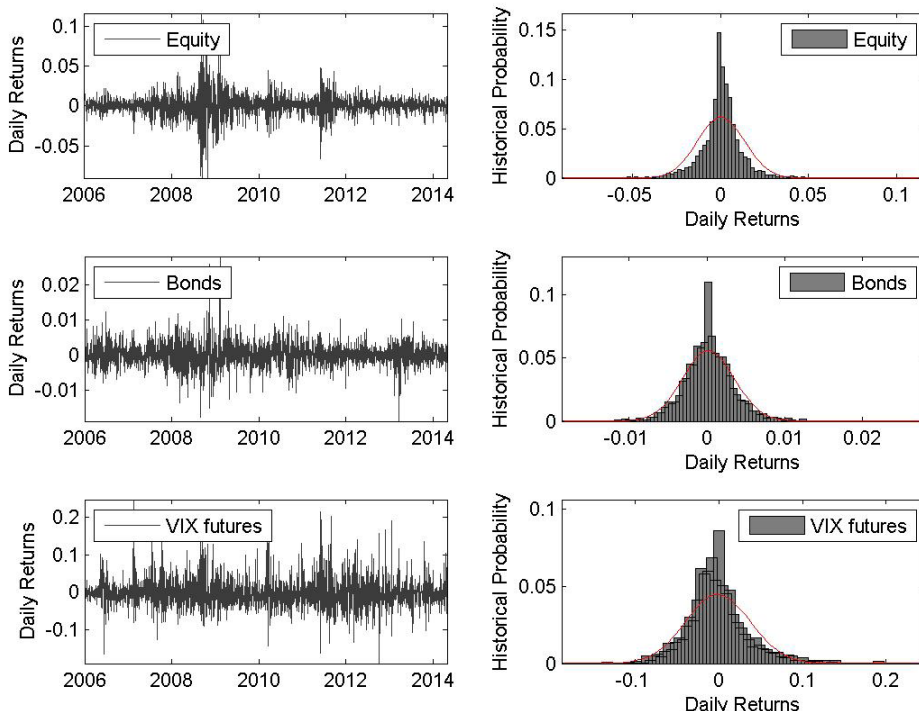
	Correlations		
	Equity	Bonds	VIX Futures
Panel A: All observations			
Equity	1	-0,08	-0,76
Bonds		1	0,06
VIX Futures			1
Panel B: Stock market crises			
Equity	1	-0,17	-0,83
Bonds		1	0,02
VIX Futures			1
Panel C: Stock market calm			
Equity	1	-0,07	-0,77
Bonds		1	0,07
VIX Futures			1

This table reports the cross-asset correlations between equity, bonds, and VIX futures, from December 30, 2005 to July 4, 2014. Calculations above are based on daily simple asset returns. Stock market crises are identified as periods of stock market turmoil, ranging from August 29, 2008 to November 11, 2008, i.e. subprime crisis, and from July 11, 2011 to October 3, 2011, i.e. European sovereign debt crisis.

Furthermore, the distinct empirical properties exhibited above raises the following statistical issue. VIX futures behave very differently from traditional asset classes, e.g. equities and bonds, as they displayed on average strongly negative returns and high volatility from December 30, 2005 to July 4, 2014. This stems from the fact that VIX futures contracts form distinct securities that do not generate certain cash flows. Although equities and

bonds pay certain dividends and coupons, respectively, VIX futures contracts provide an insurance premium against stock market crashes. Consequently, commonly used parametric methods, such as mean-variance frameworks, are inappropriate to explore VIX futures positioning within an equity-bond allocation. Therefore, it is especially true as the Figure 2 displays distinct stylized effects characterizing returns on VIX futures, in terms of higher-order moments. As expected in Table 1 and Figure 2, returns distributions of equity, bonds, and VIX futures are significantly non-normal, asymmetric, peaked and heavy-tailed. However, distinct empirical properties characterize VIX futures returns in terms of higher-order moments. From Table 1 (Panel A), returns distribution of VIX futures is strongly skewed to the right, more rounded, and less heavy-tailed ($Sk = 0.9$, $k = 6.9$), in comparison to equities ($Sk = -0.1$, $k = 13.8$), and bonds ($Sk = 0.3$, $k = 8.4$). Following Cont (2001), mean-variance approaches are invalidated when stylized effects cannot be modeled appropriately with the first two moments.

Figure 2: Historical Asset Returns Distributions



This figure displays the historical returns on equity, bonds, and VIX

futures, from December 30, 2005 to July 4, 2014. Calculations above are based on daily simple asset returns. Figures at the left exhibit time-varying historical returns. Figures at the right exhibit historical returns distributions compared to their associated normal probability density function.

This is especially the case in this paper, when considering sophisticated instruments such as volatility derivatives. Therefore, VIX futures investing would be always penalized by quadratic-preferences agents, whereas these securities offer efficient equity diversification in times of stock market turmoil. This validates the approach of direct numerical optimization undermining this paper, as it appropriately handles higher-order moments.

III. EMPIRICAL METHODOLOGY

In this section, we describe the methodology implemented to evaluate empirically the appropriateness of VIX futures to investor's risk-aversion degree. For this purpose, we expose the asset-allocation problem that consists in maximizing the agent's expected utility by optimally allocating wealth between equity, bonds, and VIX futures contracts. Subsequently, this section defines two relevant criterion measures to investigate portfolio-choice optimality under uncertainty. First, our investor welfare criterion metric measures the ex post discrepancies between the realized and expected utilities derived from VIX futures positioning. Second, the model-implied risk premium, extracted from optimal portfolios under the EU theory, evaluates portfolio insurance provided either by traditional asset classes or by VIX futures diversification.

1. FRAMEWORK

In the literature, the mean-variance framework introduced by Markowitz (1959) is one of the most commonly used approaches to examine diversification benefits. More specifically, Alexander and Korovilas (2011) use the mean-variance criterion to examine portfolio diversification with buy-and-hold positions in VIX futures contracts. However, this results in maximizing the investor's expected utility at only order two. Therefore, it does not handle higher-order moments that must be taken in account when investing in alternative assets, characterized by strongly non-normally distributed returns and substantial downside tail risk. Subsequently, the following framework that we propose proves to handle more appropriately risk preferences, especially when investing in sophisticated derivatives such as VIX futures contracts.

Pioneered by Samuelson (1969) and Merton (1969), the asset-allocation problem we apply in this paper is one of the classic problems of modern finance.

Standard formulation consists in an investor's objective to maximize the expected utility $E[U(W_T)]$ of end-of-period wealth W_T , by allocating wealth W_{T-1} at time $T-1$ between equities, bonds and VIX futures over the investment period $[T-1, T]$. We assume that his utility function $U(\cdot)$ exhibits a constant relative risk aversion, as defined below by the so-called isoelastic utility:

$$U(W_T) \equiv \frac{W_T^{1-\gamma}}{1-\gamma}, \gamma \neq 1, \gamma > 0 \quad (1)$$

where γ denotes the coefficient of agent's relative risk aversion. This results in finding the optimal investment policy $\{\omega_{i,T}^*\}$ for $i \in \{1, 3\}$, i.e. the optimal weights of equities, bonds, and VIX futures, respectively, maximizing the following expected utility over the investment period $[T-1, T]$:

$$E^P \left[U(W_T^*) \right] = \max_{\{\omega_{i,T}\}} E^P \left[U(W_T) \right] \quad (2)$$

subject to the following constraints

$$W_T = W_0 \left(1 + \sum_{i=1}^3 \omega_{i,T} r_{i,T} \right) \quad (3)$$

$$\sum_{i=1}^2 \omega_{i,T} = 1 \quad (4)$$

$$\omega_{i,T,\min} \leq \omega_{i,T} \leq \omega_{i,T,\max}, i = \{1, 2, 3\} \quad (5)$$

where

$$W_T^* = W_0 \left(1 + \sum_{i=1}^3 \omega_{i,T}^* r_{i,T} \right)$$

defining the end-of-period wealth generated by the optimal investment policy $\{\omega_{i,T}^*\}$. For $i = \{1, 2, 3\}$, $r_{i,t}$ designate respectively returns on equities, bonds and VIX futures contracts over the investment period $[T-1, T]$. $E^P [\cdot]$ refers to the expectation operator under the real-world probability measure P . As specified in (5), an optimal positioning in VIX futures contracts denoted $\omega_{3,T}^*$ provides diversification for the equity-bond portfolio defined by (4).

As follows, we describe the practical implementation of the asset-allocation problem from (2) to (4). The portfolio strategy we implement addresses practical issues frequently met within the asset management industry,

especially mitigating portfolio rebalancing without information loss. Therefore, we propose a monthly rebalancing frequency based on intra-monthly data to perform a direct numerical optimization. The budget constraints defined below allow for portfolio leverage and short sales on VIX futures contracts:

$$\begin{aligned}\omega_{i,T,\min} &= 20\%, \omega_{i,T,\max} = 80\%, i \leq 2, \\ \omega_{3,T,\min} &= -20\%, \omega_{3,T,\max} = 20\%\end{aligned}\tag{6}$$

At the end of period $[T - 1, T]$, portfolio weights solving (2)–(6) define the optimal investment policy $\{\omega_{i,T}^*\}$ used to rebalance portfolio over the period $[T, T + 1]$. Portfolio strategy at time T is then implemented as below, giving the following end-of-period portfolio value W_{T+1} at time $T + 1$

$$\tilde{W}_{T+1} = \tilde{W}_T \left(1 + \sum_{i=1}^3 \omega_{i,T}^* r_{i,T+1} \right)\tag{7}$$

where W_{T+1} defines the end-of-period wealth generated by optimal investment policy $\{\omega_{i,T}^*\}$ calculated over $[T - 1, T]$. The procedure is repeated at the end of each investment period, at equally spaced time intervals.

As mentioned earlier, one of the main practical issues undermining the investment policy specified in (7) consists in mitigating the frequency of portfolio rebalancing, as trading could be costly. However, extending the length T of investment periods is statistically detrimental for portfolio optimizations. For example, information loss generated by a monthly data frequency results from insufficient observations, i.e. nearly 40 monthly returns with only one new observation at each optimization. Therefore, we address this practical issue by using intra-monthly data, i.e. 30 daily historical returns, to perform monthly portfolio rebalancing.

Furthermore, the mean-variance criterion introduced by Markowitz is equivalent to maximizing the expected utility at order two. This approach inappropriately investigates the diversification effects provided by sophisticated instruments. Therefore, we extend the previous works of Alexander and Korovilas (2011) by using a maximization of the expected utility at higher-order moments to better handle risk preferences. As Equation (2) can't be solved exactly, Jondeau and Rockinger (2006) apply a Taylor series expansion for $U(W_T)$ of order four around $W_T = E^{\text{IP}}[W_T]$. We could use the specification (1) of the isoelastic utility function $U(\cdot)$ to obtain the approximate solution exposed in the Appendix A. However, this paper rather proposes a direct numerical optimization, where the nonlinear

programming problem (2)–(6) is solved with an active-set algorithm, up to the precision associated to termination conditions.

More specifically, the numerical optimization is based on the estimation of historical joint distributions by simulating scenarios of cross-sectional asset returns. To this purpose, we perform a multivariate block bootstrap procedure to estimate numerous trajectories of terminal wealth W_T over each investment horizon T . As described by Kunsch (1989), the block-bootstrap procedure preserves the dependence structure of asset returns, both in time and cross-sectionally. For each subsample of historical data, fixed-length blocks of cross-sectional returns are selected randomly with replacement, and then put together in a non-overlapping way to simulate a new subsample. The bootstrap procedure is repeated 10^5 times, for 30-day³ subsamples and 5-day blocks.

Finally, the benchmark used to investigate rational investment decisions is associated to the optimal portfolio solving (2)–(6) and implementation (7) with $i = \{1,2\}$, i.e. the optimal equity-bond portfolio excluding VIX futures investing.

2. WELFARE CRITERION MEASURE

Under the standard expected utility theory, decision-makers are assumed to be entirely rational machine men, devoid of anticipatory feelings, i.e. positive surprise or disappointment, when facing uncertainty. However, this assumption has been contradicted by behavioral finance theory. As in Akerlof and Dickens (1982), rational agents make decisions under risk to maximize their welfare, by anticipating the future and forming endogenous beliefs based on their preferences. Furthermore, ex ante welfare provided by anticipatory feelings is gauged by the expected future utility based on the estimation of risk distribution. Individuals feel therefore pain or disappointment if the final outcome does not reach their rational expectations about the future. By analogy, asset managers usually compare ex post the received payoff of the lottery to the anticipated payoff derived by their forecasting models. This issue is directly related to model risk management defined by Rebonato (2001) that consists in controlling discrepancies between the mark-to-model value of a security, and the market price at which it had been traded. Therefore, asset managers form ex post either pain or pleasure from the comparison between their model price and the market price.

In this paper, asset-allocation problem (2) consists of an investor's objective to maximize expected utility $E^P[U(W_T)]$ of his end-of-period wealth

3. Alternative settings for investment-time windows and block lengths are available upon request. Empirical results associated to alternative settings ensure robustness towards the conclusions exposed in this paper.

W_T . We assume that rational investors are expected-utility maximizers who compare at end-of-period T the realized utility $U(W_T)$ with the anticipated utility $E^P[U(W_T)|I_{T-1}]$ derived from implementation (7). Therefore, they form either positive or negative surprise denoted $Surprise_T$ as below

$$Surprise_T = \underbrace{U(\tilde{W}_T)}_{\substack{\text{Ex post welfare,} \\ \text{i.e. realized utility}}}} - \underbrace{E^P[U(\tilde{W}_T)|I_{T-1}]}_{\substack{\text{Ex ante welfare,} \\ \text{i.e. expected utility}}} \quad (8)$$

i.e. ex post pleasure if $Surprise_T > 0$, and ex post pain otherwise, where

$$\tilde{W}_T = \tilde{W}_{T-1} \left(1 + \sum_{i=1}^3 \omega_{i,T-1}^* r_{i,T} \right) \text{ as specified by equation (7),}$$

$$\text{and } \{I(t)\}_{t \in [0, T-1]} = \{r_{i, T-1}\} \text{ for } i \in \{1, 3\}. E^P[\cdot | \cdot]$$

is the conditional expectation operator under the real probability measure P . W_T defines the end-of-period wealth generated by optimal investment policy $\{\omega_{i, T-1}^*\}$, i.e. portfolio weights solving asset-allocation problem (2) over investment horizon $[T - 2, T - 1]$. For isoelastic functions specified in (1), negative utility requires to compare realized with anticipated utility in absolute terms. Therefore, for positive $Surprise_T$, investors feel satisfaction as the final outcome they obtained at end-of-period T exceeds their rational expectations. Conversely, negative $Surprise_T$ corresponds to unfulfilled expectations, generating investor's ex post pain.

The intuition behind our welfare criterion measure is that, when investing in VIX futures, risk-loving investors tend to feel more ex post pain than experimented and rational investors. Distorted by gambling attitudes, the decision process of risk-loving investors anticipates inappropriately the risks inherent in complex derivatives. In the literature, recent studies consistent with Whaley (2013) document the ex post welfare costs of risk-loving investors. Therefore, the hypothesis we test stipulates that our utility criterion measure $Surprise_T$ significantly improves when diversifying with VIX futures, and especially when the degree of investor's risk-aversion γ increases.

3. MODEL-IMPLIED RISK PREMIUM

Although equities and bonds generate cash flows with certainty, i.e. either dividends or coupons, VIX futures contracts provide portfolio insurance against stock market crashes. Therefore, the costs of carry consist in financing

an insurance premium to hedge portfolios against stock market downside, as investors consent to pay a risk premium to avoid uncertainty. Subsequently, the hypothesis undermining this section stipulates that VIX futures contracts provide more efficient insurance portfolio than traditionally diversified portfolios, i.e. equity-bonds portfolios.

In preference theory, risk-averse decision-makers systematically prefer to exchange a risky lottery for a certain payment, under uncertainty. Described by Mas-Colell, Whinston, and Green (1995), the risk premium defines the maximum amount of money that the risk-averse agent consents to pay to avoid lotteries riskiness. Therefore, the risk premium π_T realized at time T corresponds to the amount of money between the maximum expected wealth $E^P [W_T^*]$ and the certainty equivalent C_T :

$$\Pi_T = E^{IP} [W_T^*] - C_T \quad (9)$$

where $W_T^* = W_0 \left(1 + \sum_{i=1}^3 \omega_{i,T}^* r_{i,T} \right)$, i.e. wealth generated by optimal investment policy $\{\omega_{i,T}^*\}$.

Explicitly, equation (9) refers to the additional incentive that risk-averse agents need to take on the risk of the lottery. As specified by the equation below, the certainty equivalent C_T associated to the asset-allocation problem (2) defines the lowest amount of money received with certainty at time T for which the rational decision-maker remains indifferent to a lottery.

$$C_T = U^{-1} \left[E^P \left[U(W_T^*) \right] \right] \quad (10)$$

where $U^{-1}[\cdot]$ denotes the inverse function of the agent's utility, γ denotes the relative risk-aversion coefficient of the risk-averse agent, and $E^P [U(W_T^*)] \equiv \max_{\{\omega_{i,T}\}} E^P [U(W_T)]$ as specified by the standard allocation problem (2). Therefore, by plugging the Equation (1) of the isoelastic utility function and developing the equation of the certainty equivalent C_T , the risk premium Π_T becomes

$$\Pi_T = \underbrace{E^{IP} [W_T^*]}_{\text{Expected wealth}} - \underbrace{\left[(1-\gamma) \cdot \max_{\{\omega_{i,T}\}} E^P [U(W_T)] \right]^{\gamma-1}}_{\text{Certainty equivalent } C_T} \quad (11)$$

where $E^P [W_T^*]$ designates the maximum expected value, i.e. the objective maximum value related to the lottery, and $E^P [U(W_T)]$ is the expected utility under the real-probability measure, i.e. the subjective value related to the lottery.

Specifically, Equation (11) corresponds to the observation that risk-averse agents usually spend money to get rid of a specific risk. Risk-averse agents may like risky lotteries under uncertainty if the expected payoffs that they yield are worth the riskiness. Similarly, risk-averse investors may purchase risky assets if their expected returns exceed the risk-free rate. Theoretically, higher lotteries uncertainty and/or higher agent's degree of risk-aversion γ increase the risk premium paid to insure portfolios. Furthermore, another direct consequence is that π_T proves to be nonnegative, when $U(\cdot)$ is concave, i.e. for risk-averse agents. In accord with equation (11), we evaluate the model-implied insurance portfolio provided either by traditional asset classes, e.g. equities and bonds, or by VIX futures contracts, when portfolios are optimally allocated. In our intuition, traditional portfolios provide significantly lower model-implied risk premia π_T than alternative portfolios, i.e. overlay portfolios including VIX futures. This results that VIX futures positioning provides better portfolio insurance and higher incentives to take on the stock market risks, than traditional asset classes.

IV. EMPIRICAL RESULTS

This section examines the empirical patterns related to portfolio choice optimality, under the three performance criteria described previously. In particular, this part tests the appropriateness of VIX futures contracts to investor's risk aversion. For this purpose, optimal portfolios are investigated, first, under the criterion of risk-adjusted portfolio performance measures, handling appropriately higher-order moments; second, under the criterion of our welfare measure; and third, under the criterion of the model-implied risk premium gauging the portfolio insurance offered by optimally diversified portfolios.

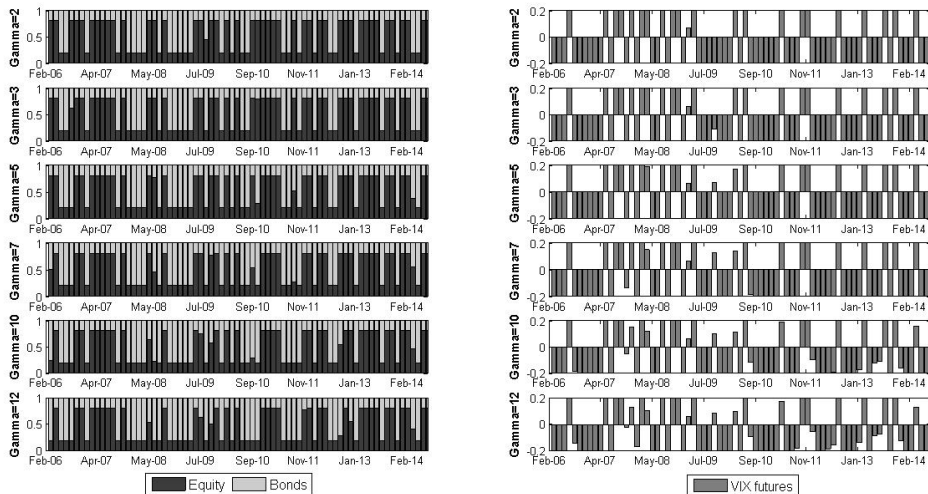
1. RISK-ADJUSTED PERFORMANCE MEASURES

This subsection compares optimal portfolios under the criterion of risk-adjusted portfolio performance measures that appropriately take into account the risks inherent to VIX futures contracts. Therefore, comparisons are twofold: on the one hand, between traditional portfolios and overlay portfolios diversified with VIX futures, both in-sample portfolios and implemented portfolios; on the other hand, between portfolios adding VIX futures in function of the degree of risk aversion.

Figure 3 exhibits the optimal investment policy $\{\omega_{i,T}^*\}$ solving the asset allocation problem (2)–(6), for an overlay portfolio composed of equities, bonds, and VIX futures contracts. Optimal portfolio weights $\{\omega_{i,T}^*\}$ clearly

exhibit time-dependency and specific cross-asset relations. Although optimal weights $\{\omega_{3,T}^*\}$ allocated to VIX futures tend to be negative on average over the entire dataset, they turn notably positive in times of stock market crashes, especially during the subprime crisis, i.e. from August 29, 2008 to November 20, 2008, and during the European sovereign debt crisis, i.e. from July 11, 2011 to October 3, 2011. Time-dependency of optimal portfolio weights $\{\omega_{i,T}^*\}$ entails in particular time-variable returns distributions for the implemented portfolios $W_{T'}$ as illustrated by Figure 4 that breaks down returns distributions into different time-periods, especially the subprime crisis and the European sovereign debt crisis. Therefore, from Figure 3, $\omega_{3,T}^* < 0$ generally implies higher $\omega_{1,T}^*$ and vice versa, capturing the inverse relation between the stock index and its implied volatility, i.e. the implied leverage effect. Furthermore, optimal portfolio weights $\{\omega_{i,T}^*\}$ depend on the coefficient γ of relative risk aversion. Increasing the risk-aversion coefficient from $\gamma = 2$ to $\gamma = 12$ mitigates portfolio overweighting and underdiversification. This result consistently follows the Modern Portfolio Theory (MPT) pioneered by Markowitz (1959), stipulating that portfolio diversification provides risk reduction. In particular, more risk-averse investors typically spread more nonsystematic risk across asset classes.

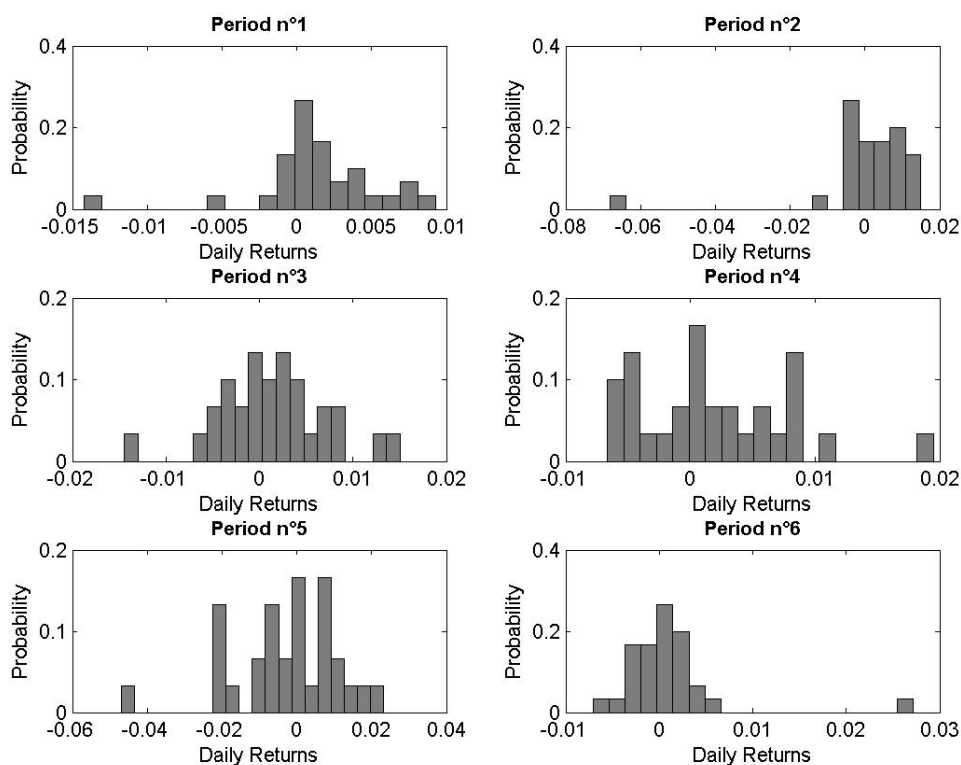
Figure 3: Optimal Weights



This figure displays the optimal weights for portfolios composed of equity,

bonds, and VIX futures, at each investment period, for different coefficients γ of relative risk aversion, from December 30, 2005 to July 4, 2014. Optimal weights correspond to the optimal investment policy $\{\omega_{i,T}^*\}$ solving asset-allocation problem (2)–(6).

Figure 4: Returns Distributions of Implemented Portfolios for Different Periods



This figure displays the returns distributions of implemented portfolios, for an asset-allocation composed of equity, bonds, and VIX futures, for different periods, and for a constant relative risk-aversion coefficient. Implemented portfolios denoted W_T are derived from equation (7). The constant coefficient of relative risk aversion corresponds to $\gamma = 5$. Period 2 and 4 are picked of stock market crises, respectively from October 3, 2008 to November 14, 2008 during the subprime crisis, and from August 19, 2011 to September 30, 2011 during the European sovereign debt crisis. In comparison, the figure also displays standard sub-periods: period 1

(from April 19, 2007 to May 30, 2007), period 3 (from September 4, 2009 to October 16, 2009), period 5 (from May 13, 2013 to June 21, 2013), and period 6 (from February 28, 2014 to April 11, 2014).

Following the statistical issue raised by the framework, commonly-used portfolio performance measures, specifically the Sharpe ratio SR , prove to be only valid for quadratic preferences. This is the case for either quadratic utility functions, and/or normally distributed asset returns, e.g. when asset returns can be precisely modelled with the first two moments.

$$SR_T = \frac{R_T - r_{f,T}}{\sigma_T} \quad (12)$$

where R_T and $r_{f,T}$ respectively refers to logarithmic returns on the portfolio and on the risk-free asset over the period $[T - 1, T]$. σ_T denotes the standard deviation of portfolio logarithmic returns. Consequently, the Sharpe ratio SR does not handle appropriately the properties inherent to sophisticated derivatives, e.g. risk preferences for higher-order moments and strongly non-Gaussian returns distributions. Therefore, we examine portfolio performances under the Adjusted for Skewness Sharpe Ratio $ASSR$, proposed by Koekebakker and Zakamouline (2009),

$$ASSR_T = SR_T \cdot \left[1 + \frac{1}{3} \left(1 + \frac{1}{\gamma} \right) Sk_T \times SR_T \right]^{1/2} \quad (13)$$

where SR_T and Sk_T refers respectively to the Sharpe ratio and to the skewness of portfolio returns distribution. As specified by (13), the Adjusted for Skewness Sharpe Ratio $ASSR$ handles investors' risk preferences at order three, penalizing high third order-moment Sk_T , especially when relative risk aversion γ increases.

Table 3 reports the risk-adjusted performance measures related, on the one hand, to the optimal portfolios W_T^* (Panel A), and on the other hand, to the implemented portfolios W_T (Panel B). Empirical results suggest that adding VIX futures to traditional equity-bond allocations significantly improves the risk-adjusted performance measures, both in-sample (Panel A) and following the implementation (Panel B). On the one hand, in the case of optimal portfolios W_T^* (Panel A), in-sample performances are calculated with the optimal investment policy $\{\omega_{i,T}^*\}$, solving portfolio problem (2)–(6) over $[T - 1, T]$. Compared to traditional portfolios (at the left), the Adjusted for Skewness Sharpe Ratio $ASSR$ of portfolios diversified with VIX futures (at the right) significantly outperforms (4.09 versus 1.93 for $\gamma = 7$). Although annualized volatility of

alternative portfolios (at the right) is higher (16.94% versus 10.10% for $\gamma = 5$), annualized return significantly outperforms (45.32% versus 16.49% for $\gamma = 5$). Besides, adding VIX futures drastically mitigates maximum drawdown (8.96% versus 13.70% for $\gamma = 10$), whereas returns distribution is left-skewed, more rounded, and less heavy-tailed ($Sk = -0.22$, $Sk = 7.68$ versus $Sk = 0.26$, $Sk = 9.10$, for $\gamma = 7$). On the other hand, in the case of implemented portfolios W_T (Panel B) globally preserves the patterns described above. Performances are calculated with optimal weights $\{\omega_{i,T}^*\}$ solving problem (2)–(6) over $[T - 1, T]$, and implemented over $[T, T + 1]$ as specified by (7). In comparison to traditional portfolios (at the left), the Adjusted for Skewness Sharpe Ratio *ASSR* of portfolios diversified with VIX futures (at the right) keeps outperforming (0.24 versus 0.14 for $\gamma = 7$). Although annualized volatility of alternative portfolios (at the right) remains higher (18.97% versus 11.32% for $\gamma = 5$), annualized return proves to significantly outperform (8.95% versus 5.48% for $\gamma = 5$). These empirical results are relevant with Moran and Dash (2007), or Brière, Burgues and Ombretta (2010). They consistently validate the robust portfolio risk-reduction and downside-risk controlling provided by VIX futures optimal positioning.

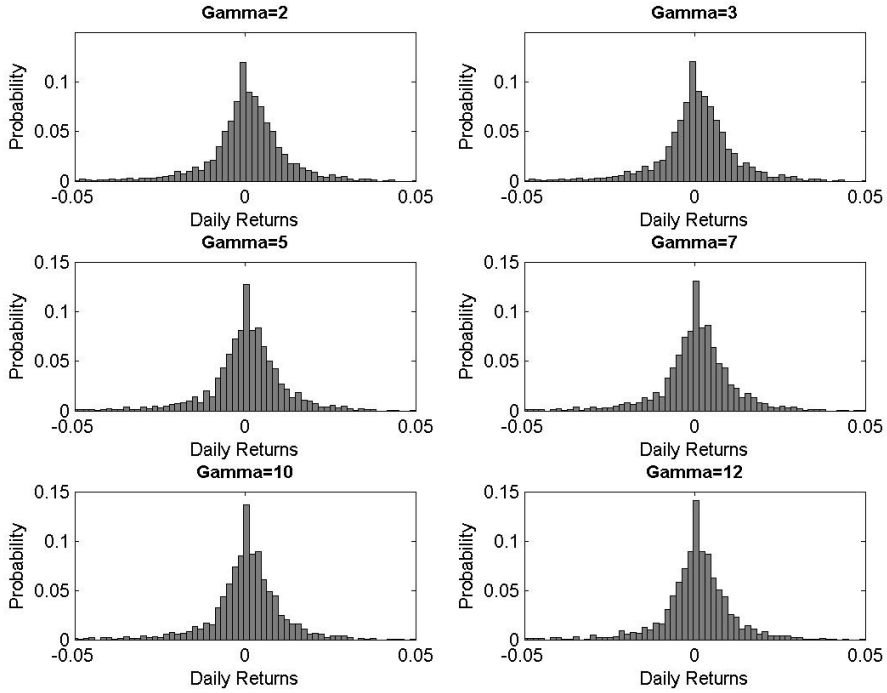
Table 3: Summary Statistics of Portfolio Performances

	Portfolio Choice without VIX Futures						Portfolio Choice with VIX Futures					
	Y = 2	Y = 3	Y = 5	Y = 7	Y = 10	Y = 12	Y = 2	Y = 3	Y = 5	Y = 7	Y = 10	Y = 12
Panel A: Optimal in-sample portfolios												
Annualized return	16,46%	16,49%	16,49%	16,34%	16,20%	16,05%	45,49%	45,45%	45,32%	45,14%	44,24%	43,52%
Annualized volatility	10,37%	10,22%	10,10%	9,77%	9,50%	9,31%	17,45%	17,30%	16,94%	16,55%	15,24%	14,48%
Skewness	0,14	0,17	0,20	0,26	0,32	0,31	- 0,24	- 0,23	- 0,23	- 0,22	- 0,07	- 0,01
Kurtosis	8,32	8,53	8,77	9,10	9,61	9,81	7,02	7,14	7,45	7,68	6,63	6,62
Maximum drawdown	13,70%	13,70%	13,70%	13,70%	13,70%	13,70%	10,87%	10,87%	10,87%	10,87%	8,96%	8,12%
Sharpe ratio	1,43	1,45	1,47	1,50	1,53	1,55	2,51	2,53	2,58	2,63	2,79	2,89
Adjusted Sharpe ratio	1,75	1,80	1,85	1,93	2,00	2,03	3,81	3,85	3,96	4,09	4,62	4,90
Panel B: Implemented portfolios												
Annualized return	5,62%	5,57%	5,48%	5,23%	5,21%	5,27%	9,23%	8,87%	8,95%	9,85%	9,91%	9,87%
Annualized volatility	11,50%	11,40%	11,32%	10,96%	10,55%	10,35%	19,61%	19,53%	18,97%	18,37%	17,62%	17,11%
Skewness	- 1,20	- 1,22	- 1,24	- 1,26	- 1,26	- 1,28	- 1,18	- 1,19	- 1,08	- 1,03	- 1,10	- 1,17
Kurtosis	15,46	15,93	16,31	16,39	16,86	17,59	12,07	12,22	10,93	10,88	11,68	12,41
Maximum drawdown	34,55%	34,55%	34,55%	34,55%	34,32%	33,38%	42,45%	42,45%	42,15%	39,50%	37,14%	36,70%
Sharpe ratio	0,35	0,34	0,34	0,33	0,34	0,35	0,39	0,37	0,38	0,45	0,47	0,48
Adjusted Sharpe ratio	0,15	0,15	0,15	0,14	0,14	0,15	0,18	0,17	0,19	0,24	0,25	0,25

This table reports the summary statistics of portfolio performances, for different coefficients of relative risk aversion, from December 30, 2005 to July 4, 2014. Optimal portfolios (Panel A), denoted $W_{T'}^*$, are derived from the optimal investment policy $\{\omega_{i,T'}^*\}$ solving the asset-allocation problem (2)–(6). Implemented portfolios (Panel B) $W_{T'}$ are derived from the allocation problem (2)–(6) and from the portfolio implementation specified by equation (7). Coefficients of relative risk aversion are denoted γ .

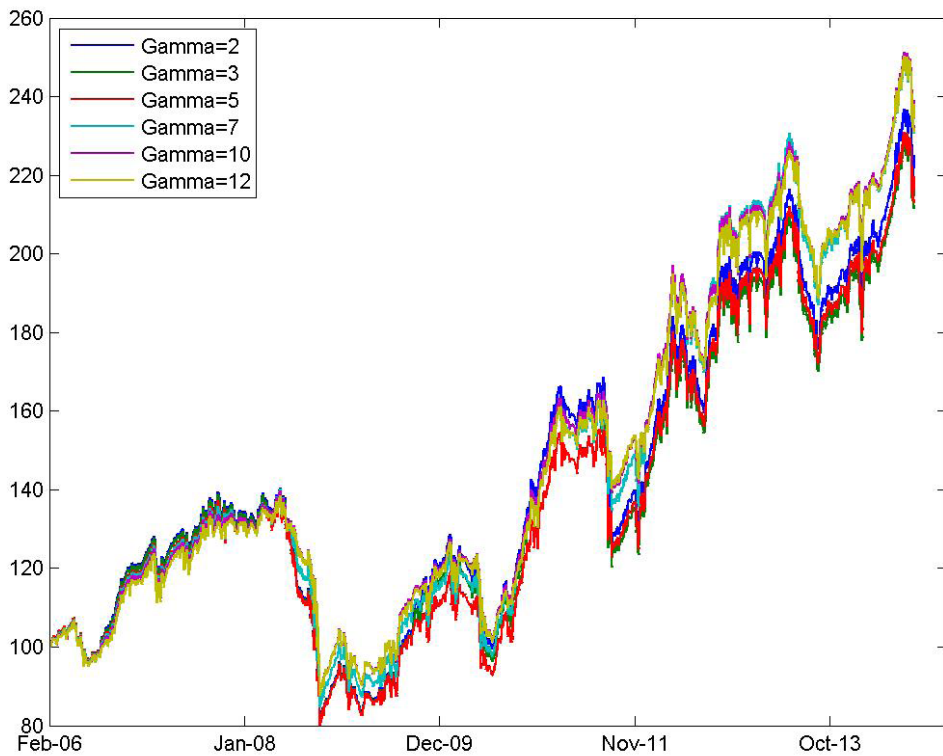
Furthermore, Table 3 and Figure 5 suggest that a higher degree of investor's risk-aversion improves notably the risk-adjusted performance measures. In particular, this is especially true for portfolios diversified with VIX futures contracts, both for portfolios W_T^* and W_T . As illustrated by Figure 5 and Figure 6 for implemented portfolios $W_{T'}$, raising the relative risk aversion γ particularly mitigates the standard deviation and generates more peaked returns distributions. More precisely, Table 3 exhibits that, when raising the coefficient of relative risk aversion from $\gamma = 2$ to $\gamma = 12$, annualized volatility decreases from 19.61% to 17.11%, whereas excess kurtosis remains high, as k ranges from 10.88 to 12.41. Therefore, this consistently mitigates the maximum drawdown from 42.45% to 36.70%. Globally, a higher risk aversion degree significantly improves the risk-adjusted performance measures of implemented portfolios $W_{T'}$, as the Adjusted for Skewness Sharpe ratio $ASSR_{T'}$ strongly increases from 0.18 to 0.25. In comparison to traditional portfolios, the Adjusted for Skewness Sharpe ratio remains stable near 0.15. However, for less risk-averse investors, adding VIX futures proves to be detrimental in terms of risk-adjusted performance measures. For $\gamma = 2$ and $\gamma = 3$, alternative portfolios (respectively 0.18 and 0.17) are quite similar to traditional portfolios (both 0.15).

Figure 5: Returns Distributions of Implemented Portfolios for Different Risk Aversion Degrees



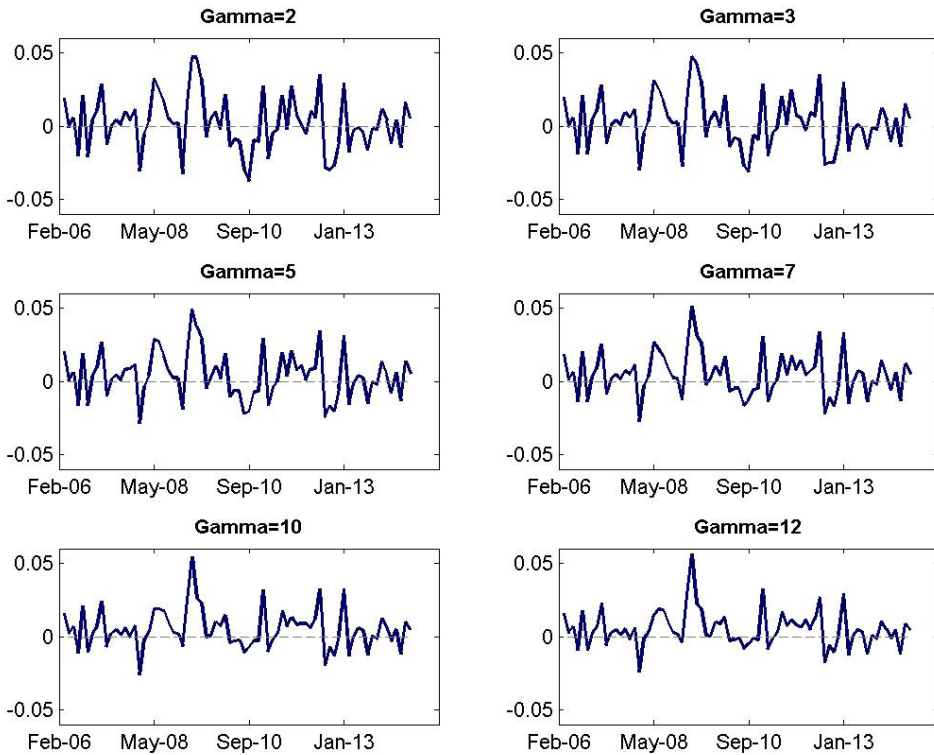
This figure displays the returns distributions of implemented portfolios, composed of equity, bonds, and VIX futures, for different coefficients of relative risk aversion, from February 13, 2006 to July 4, 2014. Implemented portfolios W_T are derived from equations (2)–(6) and implementation (7). Coefficients of relative risk aversion are denoted γ .

Figure 6: Out-of-Sample Portfolio Performances



This figure displays the evolution of implemented portfolios values, for different coefficients of relative risk aversion, from February 13, 2006 to July 4, 2014. Implemented portfolios W_T are derived from equation (7), allocated between equities, bonds, and VIX futures.

Figure 7: Investor's Surprise



This figure displays the investor's surprise formed with implemented portfolios diversified with VIX futures, for different coefficients of relative risk aversion, from February 13, 2006 to July 4, 2014. Investor's Surprise $Surprise_t$ denoted at time T is derived from (8).

Under risk-adjusted performance measures, empirical results validate the hypotheses undermining this paper. First, overlay portfolios diversified with VIX futures significantly outperforms traditionally diversified equity-bond portfolios. Second, they confirm the intuition of Whaley (2013) that VIX futures are inappropriate investments for risk-lovers and non-sophisticated investors. Furthermore, we investigate these intuitions under our welfare criterion measure *Surprise*.

2. WELFARE CRITERION MEASURE

This subsection investigates VIX futures optimal positioning under the utility criterion measure *Surprise* specified by (8). The hypotheses undermining this part are twofold. First, ex post welfare gains measured by quantity *Surprise* would be significantly higher when diversifying a traditional equity-bond portfolio with VIX futures contracts. Second, strongly risk-loving investors tend to feel more ex post pain than experimented and rational investors.

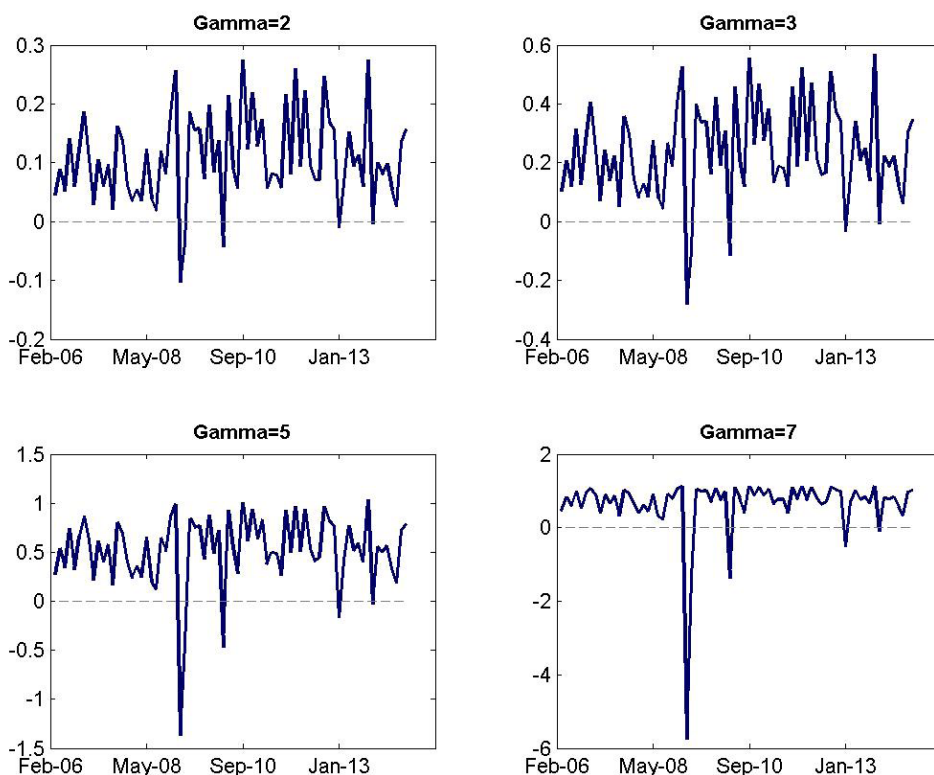
Over the entire period, Table 4 (Panel A) exhibits on average positive investor surprise *Surprise* for both traditional portfolios and portfolios adding VIX futures. Furthermore, ex post pleasure increases with risk aversion, from 0.15% to 0.39% for traditional portfolios, and from 0.17% to 0.51% for portfolios adding VIX futures. Finally, ex post elation is significantly higher when including VIX futures, whatever the risk-aversion coefficient. For example, when $\gamma = 5$, ex post positive surprise is 47% higher for portfolios adding VIX futures. These results suggest that adding VIX futures to a traditional asset allocation better exceeds rational expectations, particularly when investors are highly risk-averse, as illustrated by Figure 8. From 2005 to 2014, Table 4 breaks down investor's surprise into periods of satisfaction (Panel B) and periods of disappointment (Panel C). As defined by Equation (8), investor's satisfaction and disappointment correspond respectively to ex post positive and negative surprise *Surprise*. Although preliminary comments suggest only minor differences in the number of periods, notable differences in level of ex post elation (Panel B) or pain (Panel C) shed light on specific patterns. There are approximately as many periods of satisfaction or disappointment between the two portfolios, and across risk-aversion coefficients. For example, ex post discomfort (Panel C) ranges from 34% to 42% of total periods for traditional portfolios, and from 32% to 46% for portfolios adding VIX futures. However, satisfaction and disappointment levels are higher for portfolios including VIX futures. For example, when $\gamma = 3$, satisfaction and disappointment levels are respectively 76% (Panel B) and 78% (Panel C) higher for portfolios including VIX futures.

Table 4: Investor's Surprise

	Portfolio Choice without VIX Futures						Portfolio Choice with VIX Futures					
	Y = 2	Y = 3	Y = 5	Y = 7	Y = 10	Y = 12	Y = 2	Y = 3	Y = 5	Y = 7	Y = 10	Y = 12
Panel A: All investment horizons												
Nb investment horizons	74	74	74	74	74	74	74	74	74	74	74	74
Average	0,15%	0,19%	0,25%	0,29%	0,35%	0,39%	0,17%	0,25%	0,37%	0,46%	0,50%	0,51%
Std deviation	1,03%	1,01%	0,99%	0,97%	0,99%	1,02%	1,84%	1,74%	1,58%	1,46%	1,32%	1,25%
Panel B: Investor satisfaction												
Nb investment horizons	43	44	46	48	48	49	40	42	46	48	49	50
% investment horizons	58%	59%	62%	65%	65%	66%	54%	57%	62%	65%	66%	68%
Average	0,79%	0,80%	0,81%	0,80%	0,83%	0,84%	1,47%	1,41%	1,30%	1,25%	1,15%	1,07%
Std deviation	0,77%	0,76%	0,76%	0,78%	0,85%	0,91%	1,27%	1,23%	1,17%	1,12%	1,09%	1,07%
Panel C: Investor disappointment												
Nb investment horizons	31	30	28	26	26	25	34	32	28	26	25	24
% investment horizons	42%	41%	38%	35%	35%	34%	46%	43%	38%	35%	34%	32%
Average	-0,75%	-0,71%	-0,66%	-0,63%	-	-0,50%	-1,35%	-1,27%	-1,15%	-1,00%	-	-0,67%
Std deviation	0,57%	0,55%	0,52%	0,49%	0,48%	0,49%	1,08%	0,98%	0,79%	0,68%	0,63%	0,59%

This table reports investor's surprise formed with traditional equity-bonds portfolios (at the left), and with overlay portfolios including VIX futures (at the right), for different coefficients of relative risk aversion, from February 13, 2006 to July 4, 2014. Investor's surprise $Surprise_{\tau}$ at time T is derived from (8). Panel B investigates the periods of investor's satisfaction, i.e. ex post positive surprise. Panel C investigates the periods of investor's disappointment, i.e. ex post negative surprise. Coefficients of relative risk aversion are denoted γ .

Figure 8: Model-Implied Risk Premium



This figure displays the model-implied risk premium extracted from implemented portfolios adding VIX futures, for different coefficients of relative risk aversion, from February 13, 2006 to July 4, 2014. Model-implied risk premium, implemented portfolios values and coefficients of relative risk-aversion are denoted π_T , W_T , and γ , respectively.

Furthermore, Table 4 reports the impact of risk-aversion degree on investor's surprise when investing in VIX futures. Globally, increasing risk aversion efficiently mitigates the volatility of investor surprise and of ex post discomfort. Rising risk aversion from $\gamma = 2$ to $\gamma = 12$ reduces drastically surprise's volatility and disappointment's volatility respectively by 84% (Panel A) and by 55% (Panel C). However, for strongly risk-loving investors, adding VIX futures is detrimental in terms of welfare criterion measure *Surprise*. For $\gamma = 2$, overlay portfolios (0.17%) are nearly equivalent to traditional portfolios (0.15%), and investor's disappointment (Panel C) notably goes beyond on

average (−1.35% versus −0.75%). This strong evidence consistently extends the intuition behind the previous works of Whaley (2013) and confirms that VIX futures contracts are inappropriate buy-and-hold instruments for risk-loving investors.

3. MODEL-IMPLIED RISK PREMIUM

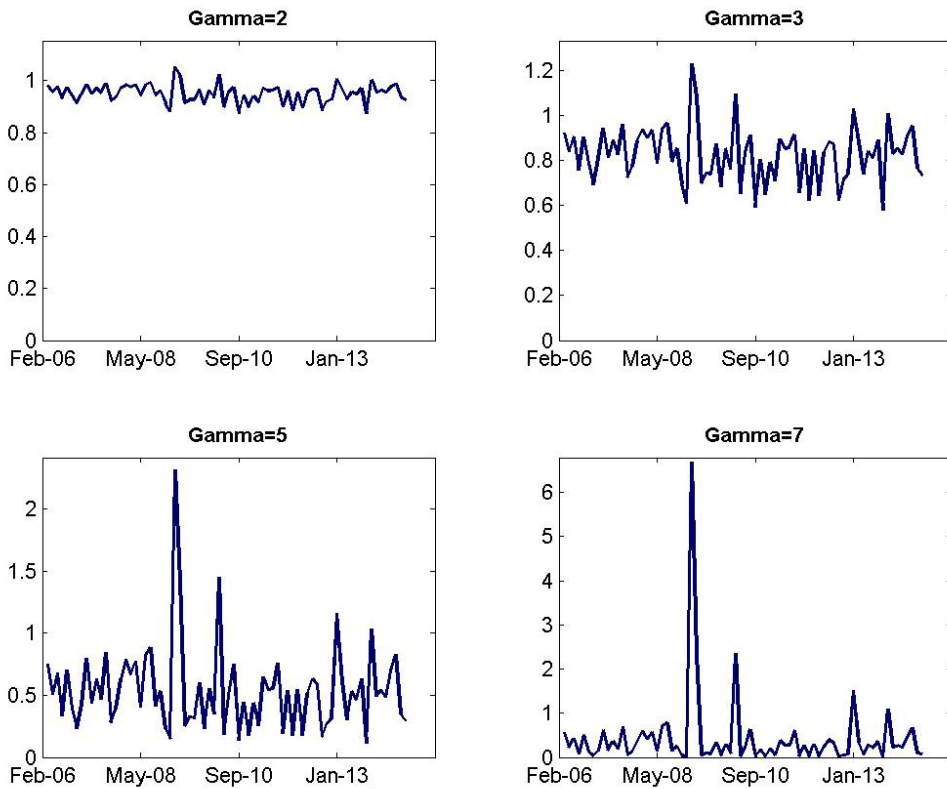
This subsection examines the level of portfolio insurance provided either by traditional portfolios, i.e. equity-bonds portfolios, or overlay portfolios, i.e. traditional portfolios diversified with VIX futures contracts. For this purpose, we estimate the risk premium π_T realized at time T , as specified by equation (11). The hypothesis undermining this part stipulates that VIX futures offer better portfolio insurance and higher incentives to take on the risks of stock market than traditional asset classes.

Table 5 reports the certainty equivalent C_T (Panel C) and the model-implied risk premium π_T (Panel D), derived from traditional portfolios (at the left) and overlay portfolios (at the right). Empirical results exhibit that for a given relative risk-aversion coefficient γ , VIX futures provide significantly higher risk premia π_T than traditional asset classes. Preliminary comments from Table 5, Figure 8, and Figure 9 are consistent with the theory of decision-making under uncertainty. First, higher coefficients γ of relative risk aversion decreases the certainty equivalent C_T and increases the risk premium π_T . For example, with regard to overlay portfolios, from $\gamma = 2$ to $\gamma = 5$, certainty equivalent C_T (Panel C) decreases from 95.01% to 54.18%, whereas risk premium π_T (Panel D) increases from 10.63% to 51.42%. This observation consistently illustrates that a more risk-averse decision-maker consents to pay higher amounts of money to avoid lotteries riskiness. Second, when relative risk-aversion $\gamma = 1$ and $\gamma > 0$, empirical results generally exhibit $\pi_T > 0$. Nevertheless, Figure 8 shows the time-varying risk premia π_T do not remain nonnegative across time. Theoretically, risk-averse agents, i.e. for concave utility function $U(\cdot)$, need an additional incentive to take on the risk of the lottery, under uncertainty. This extra incentive defines the risk premium, i.e. the cost of risk induced by lotteries uncertainty.

Table 5: Certainty Equivalent

	Portfolio Choice without VIX Futures			Portfolio Choice with VIX Futures		
	$\gamma = 2$	$\gamma = 3$	$\gamma = 5$	$\gamma = 2$	$\gamma = 3$	$\gamma = 5$
Panel A: Maximum expected utility						
Mean	-0,9824	-0,4833	-0,2351	-0,9501	-0,4529	-0,2079
Standard deviation	0,0276	0,0272	0,0266	0,0356	0,0336	0,0304
Median	-0,9817	-0,4820	-0,2328	-0,9552	-0,4573	-0,2107
Panel B: Realized utility						
Mean	-0,9809	-0,4815	-0,2325	-0,9483	-0,4504	-0,2042
Standard deviation	0,0276	0,0271	0,0263	0,0362	0,0343	0,0310
Median	-0,9808	-0,4810	-0,2314	-0,9534	-0,4547	-0,2098
Panel C: Certainty equivalent						
Mean	98,24%	93,74%	84,37%	95,01%	82,50%	54,18%
Standard deviation	0,0276	0,1060	0,4135	0,0356	0,1225	0,3435
Median	98,17%	92,94%	75,17%	95,52%	83,64%	50,41%
Panel D: Implied risk premium						
Mean	3,71%	8,21%	17,57%	10,63%	23,13%	51,42%
Standard deviation	0,0563	0,1345	0,4403	0,0761	0,1627	0,3799
Median	3,75%	8,97%	26,74%	9,50%	21,30%	54,31%

This table reports the summary statistics of certainty equivalent (Panel C) and model-implied risk premium (Panel D), for different coefficients of relative risk aversion, from February 13, 2006 to July 4, 2014. Certainty equivalent C_T and model-implied risk premium π_T are expressed as percentage of portfolio values. Coefficients of relative risk aversion are denoted γ . As comparing utility (Panels A and B) between portfolios adding VIX futures and equity-bond portfolios is irrelevant, we rather compare either certainty equivalent (Panel C), or model-implied risk premium (Panel D).

Figure 9: Certainty Equivalent

This figure displays the certainty equivalent extracted from optimal portfolios adding VIX futures, for different coefficients of relative risk aversion, from February 13, 2006 to July 4, 2014. Certainty equivalent, optimal portfolios, and coefficients of relative risk aversion are denoted C_{TV} , W_{TV} and γ , respectively.

Furthermore, according Table 5 (Panel D), VIX futures contracts provide on average significantly higher risk premia than traditional asset classes. For example, for $\gamma = 3$, alternative portfolios offer notably higher incentive (23.13%) than traditional portfolios (8.21%), for each of the lotteries. This proves particularly true for more risk-averse investors. For $\gamma = 5$, the cost of risk associated to lotteries uncertainty becomes respectively 51.42% and 17.57%. For illustration, Figure 9 exhibits the certainty equivalent for different degrees of investor's risk aversion.

This last empirical result provides twofold major findings. First, VIX futures positioning provides significantly stronger portfolio insurance

than traditionally diversified portfolios, as they better remunerate the cost of risk for each of the lotteries. Second, when investors are more risk-averse, VIX futures provide better portfolio protection than traditional asset classes, validating their appropriateness to only strongly risk-averse investors.

V. CONCLUSIONS

This paper has been motivated by the outstandingly disappointing performance of volatility derivatives. Learning from the past, dampened investors usually turn away from this original asset class, as they misunderstood risks associated to these complex instruments. Subsequently, this paper addresses the appropriateness of VIX futures contracts to investor's risk-aversion, examining portfolio-choice optimality under risk.

Empirical results provide three evidence that proved to be robust both in-sample and when implementing portfolio strategies, whatever the time settings. First, investing in VIX futures significantly beats traditionally diversified portfolios in terms of Adjusted for Skewness Sharpe Ratio and of ex post investor welfare. For example, when $\gamma = 7$, ASSR notably increases both in-sample (from 1.93 to 4.09), and when implementing portfolio strategies (from 0.14 to 0.24). Therefore, VIX futures positioning significantly improves the ex post investor welfare. When $\gamma = 7$, ex post positive surprise is on average 47% higher for portfolios adding VIX futures, suggesting that they significantly exceed investor rational expectations. Second, results confirm that VIX futures contracts are particularly inappropriate buy-and-hold instruments for strongly risk-loving investors. Increasing the relative risk-aversion from $\gamma = 2$ to $\gamma = 12$ efficiently improves in level our investor welfare metric from 0.17% to 0.51%, and drastically mitigates the volatility of investor surprise and of ex post discomfort respectively by 84% and 55%. This suggests that, when diversifying with VIX futures, risk-loving investors tend to feel more ex post pain than risk-averse investors. Third, the ex post risk premia derived from overlay portfolios, i.e. equity-bonds portfolios diversified with VIX futures, significantly outdo those derived from traditional equity-bonds portfolios. When $\gamma = 3$, VIX futures optimal positioning provides far more effective insurance premium (23.13%) than traditional equity-bonds portfolios (8.21%).

This contributes to the existing literature and opens up a range of new perspectives in the three following ways. First, our decisive contribution

validates the hypothesis undermining the previous work of Whaley (2013), i.e. VIX futures are only appropriate buy-and-hold investments for sophisticated and risk-averse investors. Therefore, this raises practical implications from the perspective of the asset management industry, as it requires intensive pedagogical efforts to educate investors about the inherent risks. Furthermore, future extensions suggest declining the exercise with mid-term VIX futures, rebalanced daily to maintain five-month constant maturity. Second, existing literature examined portfolio-choice optimality under the common mean-variance criterion, e.g. Szado (2009), Chen et al. (2011), Alexander and Korovilas (2011). Nevertheless, as stipulated by Jondeau and Rockinger (2006), the framework confirms the Markowitz (1959) approach inappropriately handles complex derivatives, under large departure from normality. To the best of our knowledge, no previous study investigated this issue under the EU framework pioneered by Samuelson (1969) and Merton (1969), for overlay allocations composed of equities, bonds, and VIX futures. Therefore, this paper proposes an alternative approach, but deep evaluations of its practicality are left for future research. Third, this paper illustrates the seminal works pioneered by Akerlof and Dickens (1982), as it derives an original welfare criterion measure to investigate the optimality of portfolio choice.

VI. REFERENCES

- G. AKERLOF and W. DICKENS. The Economic Consequences of Cognitive Dissonance. *American Economic Review*, 72(1):307-319, 1982.
- C. ALEXANDER and D. KOROVILAS. The Hazards of Volatility Diversification. Working Paper, 2011.
- M. BRIÈRE, M. BURGUES, and S. OMBRETTA. Volatility Exposure for Strategic Asset Allocation. *Journal of Portfolio Management*, 36(3):105-116, 2010.
- P. CARR and D. MADAN. Optimal Positioning in Derivative Securities. *Quantitative Finance*, 1:19-37, 2000.
- H. CHEN, S. CHUNG, and K. HO. The Diversification Effects of Volatility-Related Assets. *Journal of Banking and Finance*, 35(11):1179-1189, 2011.
- E. JONDEAU and M. ROCKINGER. Optimal Portfolio Allocation under Higher Moments. *European Financial Management*, 12(1):29-55, 2001.
- H. KUNSCH. The Jackknife and the Bootstrap for General Stationary Observations. *Annals of Statistics*, 17(3):1217-1241, 1989.

- H. MARKOWITZ. Portfolio Selection: Efficient Diversification of Investments. *Annals of Statistics*, 17(3):1217-1241, 1989.
- A. MAS-COLELL, M. WHINSTON, and J. GREEN. *Microeconomic Theory*. New York: Oxford University Press, 1995.
- R. MERTON. Lifetime Portfolio Selection under Uncertainty. *Review of Economics and Statistics*, 51(3):247-257, 1969.
- M. MORAN and S. DASH. VIX Futures and Options, Pricing and Using Volatility Products to Manage Downside Risk and Improve Efficiency in Equity Portfolios. *Journal of Trading*, 2(3):96-105, 2007.
- J. VON NEUMANN and O. MORGENSTERN. *Theory of Games and Economic Behavior*. 2nd ed. Princeton, NJ: Princeton University Press, 1995.
- R. REBONATO. Managing Model Risk. *Handbook of Risk Management*, Carol Alexander editor, FT-Prentice Hall, 2001.
- P. SAMUELSON. Lifetime Portfolio Selection by Dynamic Stochastic Programming. *Review of Economics and Statistics*, 51(3):239-246, 1969.
- W. SHARPE. Expected Utility Asset Allocation. *Financial Analysts Journal*, 63(5):1-27, 2007.
- E. SZADO. VIX Futures and Options: A Case Study of Portfolio Diversification during the 2008 Financial Crisis. *Journal of Alternative Investments*, 12(2):68-85, 2009.
- R. WHALEY. Derivatives on Market Volatility: Hedging Tools Long Overdue. *Journal of Derivatives*, 1(1):71-84, 1993.
- R. WHALEY. Understanding the VIX. *Journal of Portfolio Management*, 35(3):98-105, 2009.
- R. WHALEY. Trading Volatility: At What Cost? *Journal of Portfolio Management*, 40:95-108, 2013.
- V. ZAKAMOULINE and S. KOEKEBAKKER. Portfolio Performance Evaluation with Generalized Sharpe Ratios: Beyond the Mean and Variance. *Journal of Banking and Finance*, 33(7):1242-1254, 2009.

VII. APPENDIX

Approximate Solution for Expected Utility

In this appendix, we detail the Taylor series expansion of wealth utility $U(W_T)$, as described by Jondeau and Rockinger (2006). Applying an

approximation at order 4 around the expected wealth $E^P[W_T]$, investor's expected utility can be written as below.

$$\begin{aligned}
 E^P[U(W_T)] &\approx E^P[U(\overline{W}_T)] + E^{IP} \left[\frac{\partial U(\overline{W}_T)}{\partial W_T} (W_T - \overline{W}_T) \right] \\
 &+ \frac{1}{2!} E^P \left[\frac{\partial^2 U(\overline{W}_T)}{\partial W_T^2} (W_T - \overline{W}_T)^2 \right] + \frac{1}{3!} E^P \left[\frac{\partial^3 U(\overline{W}_T)}{\partial W_T^3} (W_T - \overline{W}_T)^3 \right] \\
 &+ \frac{1}{4!} E^P \left[\frac{\partial^4 U(\overline{W}_T)}{\partial W_T^4} (W_T - \overline{W}_T)^4 \right] + E^P[o(W_T - \overline{W}_T)]
 \end{aligned} \tag{A.1}$$

where $\overline{W}_T \equiv E^{IP}[W_T]$, and γ defines the coefficient of relative risk-aversion. Therefore, (1) for the isoelastic utility function gives the approximate solution.

$$\begin{aligned}
 E^P[U(W_T)] &\approx \frac{1}{1-\gamma} \overline{W}_T^{1-\gamma} - \frac{1}{2} E^P \left[(W_T - \overline{W}_T)^2 \right] \gamma \overline{W}_T^{-\gamma-1} \\
 &+ \frac{1}{6} E^P \left[(W_T - \overline{W}_T)^3 \right] \gamma(\gamma+1) \overline{W}_T^{-\gamma-2} \\
 &- \frac{1}{24} E^P \left[(W_T - \overline{W}_T)^4 \right] \gamma(\gamma+1)(\gamma+2) \overline{W}_T^{-\gamma-3}
 \end{aligned} \tag{A.2}$$

The 3rd and 4th terms contain risk preferences for the 3rd and 4th order of asset returns co-moments.